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State Automorphisms in Axiomatic Quantum Mechanics

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Abstract

A new metric which we call the 'intrinsic metric' is introduced on the states $\mathscr{S}$ of the generalized logic of quantum mechanics. It is shown that every automorphism on $\mathscr{S}$ is an isometry. A norm can be defined on the linear span E of $\mathscr{S}$ which reduces to the intrinsic metric on $\mathscr{S}$. If X is the completion of E then every automorphism on $\mathscr{S}$ has a unique extension to a linear isometry on X. A comparison is made between these results and those of N. Kronfli.

1. *Introduction*

In a recent paper N. Kronfli (1970) introduces a metric ρ on the set of states $\mathscr{S}$ for a quantum system which he calls the 'natural metric'. He shows that ρ can be extended to a norm on the closed linear span X of $\mathscr{S}$; and then attempts to show that any automorphism of $\mathscr{S}$ can be extended to a continuous unit normed linear operator on X. Using this last result he is able to prove the existence of Møller's wave automorphisms in abstract scattering theory (Kronfli, 1969). Unfortunately there seems to be a gap in one of these proofs which leaves the validity of these results in question.

After discussing Kronfli's results, we introduce a different metric, the 'intrinsic metric', which we feel may be of more physical significance than the natural metric. We then prove stronger results than Kronfli's in terms of the intrinsic metric.

2. *Kronfli's Results*

Following Kronfli we let $\mathscr{L}$ denote the orthomodular σ-lattice corresponding to the generalized logic of a quantum system. We denote the set of (proper) states on $\mathscr{L}$ by $\mathscr{S}$. The *natural metric* on $\mathscr{S}$ is defined by $\rho(p,q) = \sup\{|p(a) - q(a)| : a \in \mathscr{L}\}$, $p, q \in \mathscr{S}$. The proof of the following lemma is similar to part of Kronfli's Theorem 3.2 (Kronfli, 1970) except ours is more complete since we justify the interchange of the limit and summation.

Lemma 2.1: $(\mathscr{S}, \rho)$ is a complete metric space.

Proof: It is clear that $(\mathscr{S}, \rho)$ is a metric space. To show completeness let p_i be a Cauchy sequence. Then $p_i(a)$ is a real Cauchy sequence for any $a \in \mathscr{L}$ and hence converges to a number $p(a)$. Clearly $p(1) = 1$ and $p(a) \geqslant 0$ for all $a \in \mathscr{L}$. To prove countable additivity of p let a_i be a disjoint sequence in $\mathscr{L}$. Now

$$p(\vee a_i) = \lim_{j\to\infty} p_j(\vee a_i) = \lim_{j\to\infty} \lim_{n\to\infty} \sum_{i=1}^{n} p_j(a_i)$$

We now show that

$$\lim_{n\to\infty} \sum_{i=1}^{n} p_j(a_i)$$

exists uniformly in j. Let $\varepsilon > 0$. Then there is an integer N such that $i, k \geqslant N$ implies $|p_i(a) - p_k(a)| < \varepsilon/3$ for all $a \in \mathscr{L}$. Let M be an integer such that $n \geqslant M$ implies

$$\left|\sum_{i=1}^{n} p_j(a_i) - p_j(\vee a_i)\right| < \frac{\varepsilon}{3} \qquad \text{for } j = 1, 2, \ldots, N$$

Now for $n \geqslant M$, $j > N$ we have

$$\begin{aligned}\left|\sum_{i=1}^{n} p_j(a_i) - p_j(\vee a_i)\right| &\leqslant \left|p_j\left(\bigvee_1^n a_i\right) - p_N\left(\bigvee_1^n a_i\right)\right| \\ &\quad + \left|\sum_{i=1}^{n} p_N(a_i) - p_N(\vee a_i)\right| + |p_N(\vee a_i) - p_j(\vee a_i)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\end{aligned}$$

Using a standard theorem (Dunford & Schwartz, 1958, page 28), we can interchange limits to obtain

$$p(\vee a_i) = \lim_{n\to\infty} \lim_{j\to\infty} \sum_{i=1}^{n} p_j(a_i) = \sum_{i=1}^{\infty} p(a_i)$$

Thus $p \in \mathscr{S}$. To show $p_i \to p$ in the ρ metric let $k \to \infty$ in the inequality $|p_i(a) - p_k(a)| < \varepsilon/3$ for all $a \in \mathscr{L}$, $i, k \geqslant N$.

Let X_1 denote the set of signed measures on $\mathscr{L}$ and let $X \subseteq X_1$ be the closed linear span of $\mathscr{S}$. Define a norm on X_1 as follows:

$$\|p\| = \sup\{p(a) : a \in \mathscr{L}\} + \sup\{-p(a) : a \in \mathscr{L}\}$$

Kronfli shows that $(X_1, \|\cdot\|)$ is a Banach space and $\mathscr{S}$ is a closed convex subset of X_1 with the norm inducing the natural metric. The Banach space $(X, \|\cdot\|)$ is called the space of *generalized states*. If we define the partial ordering $p \leqslant q$ whenever $p(a) \leqslant q(a)$ for all $a \in \mathscr{L}$ then it is easily seen (Kronfli, 1970, Theorem 4.2) that X is an ordered Banach space with closed, normal, positive cone $C = \{p \in X: p \geqslant 0\}$. However, it is now erroneously concluded that C has nonempty interior. We now show that this need not be the case and, in fact, is never the case for most quantum systems.

The set $\mathscr{S}$ is said to be *sufficient* if for every $0 \neq a \in \mathscr{L}$ there exists $p \in \mathscr{S}$ such that $p(a) = 1$; $\mathscr{S}$ is *order determining* if $p(a) \leqslant p(b)$ for all $p \in \mathscr{S}$ implies $a \leqslant b$. In quantum mechanics $\mathscr{S}$ is usually sufficient or order determining or both (Mackey, 1963; Varadarajan, 1968; Jauch, 1968; Gudder, 1970). In particular for the usual Hilbert space quantum mechanics, $\mathscr{S}$ is sufficient and order determining. Also, except for very simple finite systems such as those in which only spin is considered, the generalized logic $\mathscr{L}$ has an infinite disjoint sequence.

Lemma 2.2: If $\mathscr{L}$ has an infinite disjoint sequence and $\mathscr{S}$ is sufficient or order determining, then the cone C in $(X, \|\cdot\|)$ has no interior points.

Proof: If $x \in C$, then x is a non-negative bounded measure on $\mathscr{L}$. If $a_i \neq 0$, $i = 1, 2, \ldots$, is an infinite disjoint sequence, since $x(\vee a_i) = \sum x(a_i) < \infty$, we must have $x(a_i) \to 0$ as $i \to \infty$. Thus for any $\varepsilon > 0$ there is $0 \neq a \in \mathscr{L}$ such that $x(a) < \varepsilon/2$. Now suppose $\mathscr{S}$ is sufficient. Then there is a $q \in \mathscr{S}$ such that $q(a) = 1$. Let $y = x - \varepsilon q$. Then $y \in X$. Now $y(a) = x(a) - \varepsilon q(a) < -\varepsilon/2$ so $y \notin C$ and yet $\|x - y\| = \|\varepsilon q\| = \varepsilon$. Thus x is not an interior point of C. Next suppose $\mathscr{S}$ is order determining. Then there is a $p \in \mathscr{S}$ such that $p(a) \geqslant \frac{1}{2}$ since if $r(a) < \frac{1}{2}$ for all $r \in \mathscr{S}$ we would have $r(a') \geqslant \frac{1}{2}$ for all $r \in \mathscr{S}$ and $a \leqslant a'$ which is impossible. Now $z = x - \varepsilon p \in X$ but $z(a) = x(a) - \varepsilon p(a) < (\varepsilon/2) - (\varepsilon/2) = 0$ so $z \notin C$. Again $\|x - z\| = \|\varepsilon p\| = \varepsilon$ so x is not an interior point of C.

Since Kronfli's Theorem 4.2 has a gap, the validity of his Theorem 4.3 which states that every $A \in \mathrm{Aut}(\mathscr{S})$ can be extended to a unit normed linear operator $\bar{A}$ on X, is unresolved. Also his Theorem 3.1 (Kronfli, 1969) which relies upon Theorem 4.3 is now unproved.

We now give a sufficient condition under which Kronfli's main results are valid. An automorphism $A \in \mathrm{Aut}(\mathscr{S})$ is *implemented* if there is an automorphism $\hat{A} \in \mathrm{Aut}(\mathscr{L})$ such that $(Ap)(a) = p(\hat{A}a)$ for all $a \in \mathscr{L}$. In the usual Hilbert space quantum mechanics every state automorphism is implemented.

Lemma 2.3: If $A \in \mathrm{Aut}(\mathscr{S})$ is implemented then A has a unique extension to an isometry of X onto itself.

Proof: We extend A by linearity to the linear span E of $\mathscr{S}$ and denote the extension by A also. Now it is easy to see that every element of E has the form $x = \alpha p - \beta q$; $\alpha, \beta \geq 0$; $p, q \in \mathscr{S}$. Now

$$\begin{aligned}\|Ax\| = \|\alpha Ap - \beta Aq\| &= \sup\{(\alpha Ap)(a) - (\beta Aq)(a) : a \in \mathscr{L}\} \\ &\quad + \sup\{(\beta Aq)(a) - (\alpha Ap)(a) : a \in \mathscr{L}\} \\ &= \sup\{\alpha p(\hat{A}a) - \beta q(\hat{A}a) : a \in \mathscr{L}\} + \sup\{\beta q(\hat{A}a) - \alpha p(\hat{A}a) : a \in \mathscr{L}\} \\ &= \sup\{\alpha p(a) - \beta q(a) : a \in \mathscr{L}\} + \sup\{\beta q(a) - \alpha p(a) : a \in \mathscr{L}\} \\ &= \|x\|\end{aligned}$$

so A is an isometry on E. The rest of the proof is now easily carried out.

Unfortunately one can give examples of state automorphisms that are not implemented so Lemma 2.3 is not universally applicable.

3. *The Intrinsic Metric*

If $p, p_1 \in \mathscr{S}$ and $0 \leq \lambda \leq 1$ then $(1-\lambda)p + \lambda p_1$ represents a mixture of p and p_1 with $(1-\lambda)$ parts p and λ parts p_1. If $p, q \in \mathscr{S}$ are 'nearly the same' one might expect to be able to obtain p from q (and q from p) by mixing q with a small amount of some other state. Conversely, if there exist p_1, $q_1 \in \mathscr{S}$ and a small number $0 < \lambda < 1$ such that $(1-\lambda)p + \lambda p_1 = (1-\lambda)q + \lambda q_1$ then p and q are close in some sense and the parameter λ is a measure of their distance apart. Motivated by the above, we define

$$\sigma(p,q) = \inf\{0 \leq \lambda \leq 1 : (1-\lambda)p + \lambda p_1 = (1-\lambda)q + \lambda q_1, p_1, q_1 \in \mathscr{S}\}$$

Notice, since $\frac{1}{2}p + \frac{1}{2}q = \frac{1}{2}q + \frac{1}{2}p$ we have $0 \leq \sigma(p,q) \leq \frac{1}{2}$. It turns out to be more convenient to make a change of scale and define the distance between p and q by $d(p,q) = \sigma(p,q)[1 - \sigma(p,q)]^{-1}$, so $0 \leq d(p,q) \leq 1$.

Lemma 3.1: σ and d are metrics on $\mathscr{S}$ and $\frac{1}{4}\rho(p,q) \leq \sigma(p,q) \leq d(p,q)$.

Proof: It is clear that σ and d are non-negative, symmetric and that $\sigma(p,p) = d(p,p) = 0$. If $\sigma(p,q) = 0$, there exist $\lambda_i \to 0$, $p_i, q_i \in \mathscr{S}$ such that $(1-\lambda_i)p + \lambda_i p_i = (1-\lambda_i)q + \lambda_i q_i$. If $a \in \mathscr{L}$, then

$$|p(a) - q(a)| \leq |\lambda_i||p(a) + q_i(a) - p_i(a) - q_i(a)| \leq 4|\lambda_i| \to 0 \qquad \text{as } i \to \infty$$

Hence $p(a) = q(a)$ and $p = q$. If $d(p,q) = 0$ then $\sigma(p,q) = 0$ so $p = q$. To prove the triangle inequality suppose

$$0 < \lambda_1, \quad \lambda_2 < 1 \quad \text{and} \quad (1-\lambda_1)p + \lambda_1 p_1 = (1-\lambda_1)s + \lambda_1 s_1$$

and

$$(1-\lambda_2)s + \lambda_2 s_2 = (1-\lambda_2)q + \lambda_2 q_1$$

Letting

$$\lambda_0 = (\lambda_1 + \lambda_2 - 2\lambda_1\lambda_2)(1 - \lambda_1\lambda_2)^{-1}$$

and

$$\lambda_3 = \lambda_2(1-\lambda_1)(\lambda_1 + \lambda_2 - 2\lambda_1\lambda_2)^{-1}$$

we see that $0 < \lambda_0, \lambda_3 < 1$ and after some algebra we find

$$(1-\lambda_0)p + \lambda_0[(1-\lambda_3)p_1 + \lambda_3 s_2] = (1-\lambda_0)q + \lambda_0[(1-\lambda_3)s_1 + \lambda_3 q_1]$$

Hence

$$\lambda_0 \in \{0 \leqslant \lambda \leqslant 1 : (1-\lambda)p + \lambda r = (1-\lambda)q + \lambda t, r, t \in \mathscr{S}\}$$

Since $\lambda_0(1-\lambda_0)^{-1} = \lambda_1(1-\lambda_1)^{-1} + \lambda_2(1-\lambda_2)^{-1}$ we have $d(p,q) \leqslant d(p,s) + d(s,q)$ for all $p, q, s \in \mathscr{S}$. The triangle inequality for σ follows in a similar way using the fact that $\lambda_0 \leqslant \lambda_1 + \lambda_2$. Finally for any $\varepsilon > 0$ there exists $0 \leqslant \lambda \leqslant 1$ and $p_1, q_1 \in \mathscr{S}$ such that $(1-\lambda)p + \lambda p_1 = (1-\lambda)q + \lambda q_1$ and $\sigma(p,q) \leqslant \lambda \leqslant \sigma(p,q) + \varepsilon$. Then for any $a \in \mathscr{L}$,

$$|p(a) - q(a)| = |\lambda p(a) + \lambda q_1(a) - \lambda p_1(a) - \lambda q(a)| \leqslant 4\lambda \leqslant 4\sigma(p,q) + 4\varepsilon$$

Hence $\rho(p,q) \leqslant 4\sigma(p,q)$.

We call d the *intrinsic metric* for $\mathscr{S}$. This metric is physically motivated and is determined by the geometric structure of the states alone without relying on the particular form the states take. As we shall see, this metric has significant regularity properties as far as automorphisms on $\mathscr{S}$ are concerned. In another paper the author plans to compare this metric with other commonly used metrics. In particular we shall show that the intrinsic metric is equivalent to the trace metric used by Jauch, Misra & Gibson (1968) in their latest study of scattering theory.

Lemma 4.2: If $A \in \mathrm{Aut}(\mathscr{S})$ then A is an isometry in the metric spaces $(\mathscr{S}, \sigma)$, $(\mathscr{S}, d)$.

Proof: For $p, q \in \mathscr{S}$ we have

$$\begin{aligned}\sigma(p,q) &= \inf\{0 \leqslant \lambda \leqslant 1 : (1-\lambda)p + \lambda p_1 = (1-\lambda)q + \lambda q_1, p_1, q_1 \in \mathscr{S}\}\\ &= \inf\{0 \leqslant \lambda \leqslant 1 : A[(1-\lambda)p + \lambda p_1] = A[(1-\lambda)q + \lambda q_1], p_1, q_1 \in \mathscr{S}\}\\ &= \inf\{0 \leqslant \lambda \leqslant 1 : (1-\lambda)Ap + \lambda Ap_1 = (1-\lambda)Aq + \lambda Aq_1, p_1, q_1 \in \mathscr{S}\}\\ &= \inf\{0 \leqslant \lambda \leqslant 1 : (1-\lambda)Ap + \lambda p_1 = (1-\lambda)Aq + \lambda q_1, p_1, q_1 \in \mathscr{S}\}\\ &= \sigma(Ap, Aq)\end{aligned}$$

It follows that $d(p,q) = d(Ap, Aq)$.

As before we let E be the linear span of $\mathscr{S}$. Now every $x \in E$ admits a representation $x = cp - dq$ where $c, d \geqslant 0, p, q \in \mathscr{S}$. Define

$$|x| = \inf\{\max(c,d) : x = cp - dq; c, d \geqslant 0; p, q \in \mathscr{S}\}$$

Theorem 3.3: $(E, |\cdot|)$ is a normed linear space such that $|p - q| = d(p,q)$ for all $p, q \in \mathscr{S}$.

Proof: Clearly $|x| \geqslant 0$, $|0| = 0$. To show $|\alpha x| = |\alpha||x|$ for all $\alpha \in R$, $x \in E$ we consider two cases. If $\alpha > 0$ then

$$\begin{aligned}|\alpha x| &= \inf\{\max(c,d) : \alpha x = cp - dq; c, d \geqslant 0; p, q \in \mathscr{S}\}\\ &= \inf\{\max(\alpha c, \alpha d) : x = cp - dq; c, d \geqslant 0; p, q \in \mathscr{S}\}\\ &= \alpha|x| = |\alpha||x|\end{aligned}$$

If $\alpha < 0$ then

$$\begin{aligned}|\alpha x| &= \inf\{\max(c,d): \alpha x = cp - dq;\ c,d \geqslant 0;\ p,q \in \mathscr{S}\}\\ &= \inf\{\max(-\alpha c, -\alpha d): x = dq - cp;\ c,d \geqslant 0;\ p,q \in \mathscr{S}\}\\ &= -\alpha|x| = |\alpha||x|\end{aligned}$$

To prove the triangle inequality, suppose $x_1, x_2 \in E$. If $x_1 = c_1 p_1 - d_1 q_1$ and $x_2 = c_2 p_2 - d_2 p_2$ then we have

$$\begin{aligned}x_1 + x_2 = (c_1 + c_2)&[c_1(c_1 + c_2)^{-1} p_1 + c_2(c_1 + c_2)^{-1} p_2]\\ &- (d_1 + d_2)[d_1(d_1 + d_2)^{-1} q_1 + d_2(d_1 + d_2)^{-1} q_2]\end{aligned}$$

where the expressions in square brackets are in $\mathscr{S}$. Then

$$\begin{aligned}|x_1 + x_2| &= \inf\{\max(c,d): x_1 + x_2 = cp - dq;\ c,d \geqslant 0;\ p,q \in \mathscr{S}\}\\ &\leqslant \inf\{\max(c_1 + c_2, d_1 + d_2): x_1 = c_1 p_1 - d_1 q_1;\\ &\qquad x_2 = c_2 p_2 - d_2 q_2\}\\ &\leqslant \inf\{\max(c_1, d_1) + \max(c_2, d_2): x_1 = c_1 p_1 - d_1 q_1,\\ &\qquad x_2 = c_2 p_2 - d_2 q_2\}\\ &= \inf\{\max(c_1, d_1): x_1 = c_1 p_1 - d_1 q_1;\ c_1 d_1 \geqslant 0,\\ &\qquad p_1 q_1 \in \mathscr{S}\} + \inf\{\max(c_2, d_2): x_2 = c_2 p_2 - d_2 q_2;\\ &\qquad c_2, d_2 \geqslant 0, p_2 q_2 \in \mathscr{S}\}\\ &= |x_1| + |x_2|\end{aligned}$$

We now show $|p - q| = d(p,q)$ for all $p, q \in \mathscr{S}$. If $p - q = cp_1 - dq_1$; $c, d \geqslant 0$; $p_1, q_1 \in \mathscr{S}$; then $0 = p(1) - q(1) = cp_1(1) - dq_1(1) = c - d$. Hence all representations of $p - q$ are of the form $p - q = c(p_1 - q_1)$ for $c \geqslant 0$; $p_1, q_1 \in \mathscr{S}$. Now

$$\begin{aligned}\sigma(p,q) &= \inf\{0 \leqslant \lambda \leqslant 1: (1-\lambda)p + \lambda p_1 = (1-\lambda)q + \lambda q_1, p_1, q_1 \in \mathscr{S}\}\\ &= \inf\{0 \leqslant \lambda \leqslant 1: p - q = \lambda(1-\lambda)^{-1}(q_1 - p_1), p_1, q_1 \in \mathscr{S}\}\\ &= \inf\{c(c+1)^{-1}: c \geqslant 0; p - q = c(q_1 - p_1); p_1, q_1 \in \mathscr{S}\}\\ &= \inf\{c \geqslant 0: p - q = c(q_1 - p_1)\}[\inf\{c \geqslant 0: p - q = c(q_1 - p_1)\} + 1]^{-1}\\ &= |p - q|[|p - q| + 1]^{-1}\end{aligned}$$

Hence $\rho(p,q) = \sigma(p,q)[1 - \sigma(p,q)]^{-1} = |p - q|$. We next show that $|p| = 1$ for all $p \in \mathscr{S}$. Indeed, if $p = cp_1 - dq_1$; $p_1, q_1 \in \mathscr{S}$; $c, d \geqslant 0$; then $1 = p(1) = c - d$. Since $d \geqslant 0$, we have $c \geqslant 1$ so $|p| \geqslant 1$. But $p = p - 0q$ so $|p| = 1$. Finally suppose $|x| = 0$ and $x = cp - dq$. Then

$$0 = |x| = |cp - dq| \geqslant ||cp| - |dq|| = |c|p| - d|q|| = |c - d|$$

so $c = d$. Hence

$$0 = |x| = c|p - q| = cd(p,q)$$

If $c \neq 0$ then $d(p,q) = 0$ giving $p = q$ and $x = 0$, if $c = 0$ then again $x = 0$.

Part of Theorem 3.3 could be proved more easily by noting that $|\cdot|$ is the Minkowshi functional (Peressini, 1967) of the convex, balanced, absorbing set $D = \{cp - dq : 0 \leqslant c, d \leqslant 1; p, q \in \mathscr{S}\} \subseteq E$. However, our proof is independent of a knowledge of Minkowshi functionals.

If we let X be the completion of the metric space $(E, |\cdot|)$ then $(X, |\cdot|)$ is the Banach space of generalized states.

Theorem 3.4: If $A \in \mathrm{Aut}(\mathscr{S})$ then A has a unique extension to a linear isometry $\tilde{A}$ on $(X, |\cdot|)$.

Proof: We show that A has a unique extension to an isometry on $(E, |\cdot|)$ and since E is dense in X the result will follow. Now any $x \in E$ admits a representation $x = cp - dq$; $c, d \geqslant 0$; $p, q \in \mathscr{S}$. Define $\tilde{A}x = cAp - dAq$. To show $\tilde{A}$ is well-defined suppose also that $x = c_1 p_1 - d_1 q_1$; $c_1, d_1 \geqslant 0$; $p_1, q_1 \in \mathscr{S}$. Then $c - d = x(1) = c_1 - d_1$ and hence (notice that $c_1 + d$, $c + d_1 > 0$ unless $x = 0$ in which case the result is trivial)

$$c(c + d_1)^{-1} p + d_1(c + d_1)^{-1} q_1 = c_1(c_1 + d)^{-1} p_1 + d(c_1 + d)^{-1} q$$

Then

$$c(c + d_1)^{-1} Ap + d_1(c + d_1)^{-1} Aq_1 = c_1(c_1 + d)^{-1} Ap_1 + d(c_1 + d) Aq$$

It follows that $cAp - dAq = c_1 Ap_1 - d_1 Aq_1$ and $\tilde{A}$ is well-defined. Finally for $x \in E$ we have

$$\begin{aligned} |x| &= \inf\{\max(c, d) : x = cp - dq; c, d \geqslant 0; p, q \in \mathscr{S}\} \\ &= \inf\{\max(c, d) : x = cA^{-1}p - dA^{-1}q; c, d \geqslant 0; p, q \in \mathscr{S}\} \\ &= \inf\{\max(c, d) : \tilde{A}x = cp - dq; c, d \geqslant 0; p, q \in \mathscr{S}\} \\ &= |\tilde{A}x| \end{aligned}$$

It follows from Lemma 3.2 or Theorem 3.4 that if we define the asymptotic condition in terms of the intrinsic metric d then Kronfli's Theorem 3.1 (Kronfli, 1969) holds for d. Since, as mentioned earlier, the intrinsic metric is equivalent to the trace metric in Hilbert space quantum mechanics, this generalizes a result of Jauch, Misra and Gibson.

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